

String theory and the Kawai-Lewellen-Tye relations between gravity and gauge theory including external matter

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We consider the Kawai-Lewellen-Tye (KLT) factorizations of gravity scalar-leg amplitudes into products of Yang-Mills scalar-leg amplitudes. We check and examine the factorizations at $\mathcal{O}(1)$ in α' and extend the analysis by considering KLT mapping in the case of generic effective Lagrangians for Yang-Mills theory and gravity.

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I. INTRODUCTION

Recently, the so-called Kawai-Lewellen-Tye (KLT) relations were reanalyzed from the perspective of effective field theory [1,2]. It was found that KLT relationships were valid for effective amplitudes, and that effective generalizations were possible. Nontrivial diagrammatic relationships between the amplitudes were also discovered presenting interesting links between the pure Yang-Mills amplitudes as well as between gravity and Yang-Mills amplitudes. It was suggested that introducing external source fields in the actions might be a way to further extend the KLT mapping. This paper aims to achieve this by including into the formalism a massless scalar field. Investigations of KLT in the presence of matter have previously been carried out in Ref. [3], although in a different setting.

The fundamental dualities of string theory, linking closed and open strings, contain also the possibility of combining seemingly completely uncorrelated field theories at low energies, i.e., below the Planck scale. String theory will at such scales *essentially* correspond to a particular version of an effective field theory, described by a local effective action where the massive particle modes are integrated out, see, e.g., Refs. [4–11]. An example of a field theory relationship induced by string theory dualities is the KLT relations first discovered in Ref. [12]. The KLT relationship represents a curious connection between two dissimilar field theories, namely, perturbative Yang-Mills theory and gravity. Being both non-Abelian gauge theories they possess some similarities in their description, but in their dynamical behavior they are quite different. For example, Yang-Mills theory is an asymptotically free theory at high energies, while gravity is well defined in the infrared but ultraviolet troubled. For example, the fundamental Yang-Mills action is renormalizable at $D = 4$, while gravity is well known to be nonrenormalizable in four dimensions, see, e.g., Refs. [13–15].

Effective field theory [16,17] provides a way to resolve the renormalization difficulties of gravity. Treated as an effective field theory the renormalization problem of the gravitational action no longer exists because all higher derivative terms possibly generated by the loop diagrams are already included in the action. Thus by each loop order all divergent loop terms can be absorbed into the coupling constants of the theory. See Refs. [18–21] for some explicit calculations involving effective gravity. To treat the Yang-Mills action by effective means is a possibility although not a necessity for renormalization reasons, at $D \leq 4$.

The KLT relations at tree level are directly linking amplitudes in Yang-Mills theory and gravity [22]. Hence useful knowledge about tree gravity scattering amplitudes can be extracted from the much simpler Yang-Mills tree amplitudes. Through cuts of loop diagrams, KLT techniques can be applied with great success in loop calculations—using propagator cuts and the unitarity of the S -matrix. As examples of such investigations, see, e.g., Refs. [23–27]. One important result shown this way was that $\mathcal{N} = 8$ SUGRA is less divergent in the ultraviolet than was previously believed to be the case. For a good review see Ref. [28].

Investigations of factorization of gravity vertices at the Lagrangian level have been carried out in Ref. [29]. In Ref. [30] rather mysterious factorizations of gravity vertices have been investigated employing a particular formalism involving vierbeins.

The structure of this paper will be as follows. We will first briefly discuss the KLT relationship, next we will consider the fundamental Lagrangians and the amplitudes in gravity and Yang-Mills theory. Through a number of examples we will directly demonstrate that the KLT relationship works for amplitudes involving scalar fields—the most challenging example being a 5-point amplitude mapping. Next the effective extension of this is discussed, and we present to lowest order the effective Lagrangians in Yang-Mills theory and in gravity including the adequate terms for the massless scalar field. Through a few basic examples we will show that the

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KLT mapping should be possible for effective amplitudes too. Finally we will summarize and discuss what has been achieved.

Throughout this paper we will use the $(+ - - -)$ metric and employ the notation $s_{12} = 2(k_1 \cdot k_2), \dots$. Helicity representations of amplitudes are employed whenever useful.¹ We put $g = \frac{1}{2}$ and $\kappa = 1$ in all calculations.

II. THEORY

From string theory [34] we know that the generic M -point amplitude of a closed string relates to a product

of open-string amplitudes in the following way:

$$\mathcal{A}_{\text{closed}}^M \sim \sum_{\Pi, \tilde{\Pi}} e^{i\pi\Phi(\Pi, \tilde{\Pi})} \mathcal{A}_M^{\text{left open}}(\Pi) \tilde{\mathcal{A}}_M^{\text{right open}}(\tilde{\Pi}), \quad (1)$$

here Π and $\tilde{\Pi}$ are the cyclic orderings associated with the external open-string right and left moving sources. The phase factor $\Phi(\Pi, \tilde{\Pi})$ of the exponential relates explicitly to the appropriate cyclic permutations of the open-string sources.

We have the following KLT relations for the 3-, 4-, and 5-point amplitudes:

$$\mathcal{M}_{3\text{gravity}}^{\mu\tilde{\mu}\nu\tilde{\nu}\rho\tilde{\rho}}(1, 2, 3) = -i\mathcal{A}_{3\text{L-gauge}}^{\mu\nu\rho}(1, 2, 3) \times \tilde{\mathcal{A}}_{3\text{R-gauge}}^{\tilde{\mu}\tilde{\nu}\tilde{\rho}}(1, 2, 3), \quad (2)$$

$$\text{and} \quad \mathcal{M}_{4\text{gravity}}^{\mu\tilde{\mu}\nu\tilde{\nu}\rho\tilde{\rho}\sigma\tilde{\sigma}}(1, 2, 3, 4) = -\frac{i}{\pi\alpha'} \sin(\pi s_{12}\alpha') [\mathcal{A}_{4\text{L-gauge}}^{\mu\nu\rho\sigma}(1, 2, 3, 4) \times \tilde{\mathcal{A}}_{4\text{R-gauge}}^{\tilde{\mu}\tilde{\nu}\tilde{\rho}\tilde{\sigma}}(1, 2, 4, 3)], \quad (3)$$

$$\begin{aligned} \mathcal{M}_{5\text{gravity}}^{\mu\tilde{\mu}\nu\tilde{\nu}\rho\tilde{\rho}\sigma\tilde{\sigma}\tau\tilde{\tau}}(1, 2, 3, 4, 5) = & -\frac{i}{\pi^2\alpha'^2} [\sin(\pi s_{12}\alpha') \sin(\pi s_{34}\alpha') (\mathcal{A}_{5\text{L-gauge}}^{\mu\nu\rho\sigma\tau}(1, 2, 3, 4, 5) \times \tilde{\mathcal{A}}_{5\text{R-gauge}}^{\tilde{\mu}\tilde{\nu}\tilde{\rho}\tilde{\sigma}\tilde{\tau}}(2, 1, 4, 3, 5)) \\ & + \sin(\pi s_{13}\alpha') \sin(\pi s_{24}\alpha') (\mathcal{A}_{5\text{L-gauge}}^{\mu\nu\rho\sigma\tau}(1, 3, 2, 4, 5) \times \tilde{\mathcal{A}}_{5\text{R-gauge}}^{\tilde{\mu}\tilde{\nu}\tilde{\rho}\tilde{\sigma}\tilde{\tau}}(3, 1, 4, 2, 5))]. \end{aligned} \quad (4)$$

In the above equations \mathcal{M} is the gravity tree amplitude, and \mathcal{A} is the color-ordered amplitude for the gauge theory. We assume identical left and right-moving theories. The explicit forms of the above KLT relations are fitted to match our conventions, which differ from those of [12].

To order $\mathcal{O}(1)$ in α' , the above relations are reduced in the following way:

$$\mathcal{M}_{3\text{gravity}}^{\mu\tilde{\mu}\nu\tilde{\nu}\rho\tilde{\rho}}(1, 2, 3) = -i\mathcal{A}_{3\text{L-gauge}}^{\mu\nu\rho}(1, 2, 3) \tilde{\mathcal{A}}_{3\text{R-gauge}}^{\tilde{\mu}\tilde{\nu}\tilde{\rho}}(1, 2, 3), \quad (5)$$

$$\mathcal{M}_{4\text{gravity}}^{\mu\tilde{\mu}\nu\tilde{\nu}\rho\tilde{\rho}\sigma\tilde{\sigma}}(1, 2, 3, 4) = -is_{12}\mathcal{A}_{4\text{L-gauge}}^{\mu\nu\rho\sigma}(1, 2, 3, 4) \tilde{\mathcal{A}}_{4\text{R-gauge}}^{\tilde{\mu}\tilde{\nu}\tilde{\rho}\tilde{\sigma}}(1, 2, 4, 3), \quad (6)$$

and

$$\begin{aligned} \mathcal{M}_{5\text{gravity}}^{\mu\tilde{\mu}\nu\tilde{\nu}\rho\tilde{\rho}\sigma\tilde{\sigma}\tau\tilde{\tau}}(1, 2, 3, 4, 5) = & -is_{12}s_{34}\mathcal{A}_{5\text{L-gauge}}^{\mu\nu\rho\sigma\tau}(1, 2, 3, 4, 5) \tilde{\mathcal{A}}_{5\text{R-gauge}}^{\tilde{\mu}\tilde{\nu}\tilde{\rho}\tilde{\sigma}\tilde{\tau}}(2, 1, 4, 3, 5) \\ & -is_{13}s_{24}\mathcal{A}_{5\text{L-gauge}}^{\mu\nu\rho\sigma\tau}(1, 3, 2, 4, 5) \tilde{\mathcal{A}}_{5\text{R-gauge}}^{\tilde{\mu}\tilde{\nu}\tilde{\rho}\tilde{\sigma}\tilde{\tau}}(3, 1, 4, 2, 5). \end{aligned} \quad (7)$$

The fundamental Einstein-Hilbert Lagrangian in gravity, including a real scalar field ϕ , reads

$$\mathcal{L} = \sqrt{-g} \left[\frac{2R}{\kappa^2} + \frac{1}{2} g^{\mu\nu} D_\mu \phi D_\nu \phi \right], \quad (8)$$

where $\kappa^2 = 32\pi G$, $g_{\mu\nu}$ is the metric field, $g = \det(g_{\mu\nu})$ and R is the scalar curvature. Similarly in Yang-Mills theory the corresponding fundamental Lagrangian is

¹The helicity representation and its twistor space [31] connection have lately caught attention initiated by the papers [32,33]. The viewpoint is essentially that perturbative gauge theory though the helicity representation can be seen as a string theory in twistor space. This might also have interesting implications for the KLT relationship.

$$\mathcal{L} = \text{tr} \left[\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} D_\mu \phi D^\mu \phi \right], \quad (9)$$

where A_μ is the vector field, ϕ is a real field and in the adjoint representation, and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g[A_\mu, A_\nu]. \quad (10)$$

The trace is over the generators of the non-Abelian Lie algebra. We normalize such that $\text{tr}(t^a t^b) = -\delta^{ab}$ and $[t^a, t^b] = -f^{abc} t^c$.

Treating gravity and gauge theory as effective theories, the Lagrangians of both theories have to be augmented with all possible higher derivative terms. For each of these terms we associate a factor of $(\alpha')^P$ where the power P is governed by the number of derivatives in the gravitational terms and by the mass dimension of the gauge terms.

Some terms in the Lagrangian may change under field redefinitions known to leave the S -matrix unchanged. We keep the minimal number of terms in the Lagrangian required to parametrize the S -matrix, and leave out the redundant ones.

III. RESULTS

A. The KLT relations to leading order in α'

To verify the KLT relations between the basic amplitudes we will consider three types of amplitude mappings, all of which involve scalar legs. We let s , v , and h represent scalars, vectors, and gravitons, respectively.

Using the above Lagrangians we extract the Feynman vertex rules and calculate the scattering amplitudes. We consider first the four scalar-leg amplitude. Here the basic one-graviton exchange diagram is related to the one-vector exchange diagram through KLT.

On the gravity side we have

$$\begin{aligned}\mathcal{M}(s_1, s_2, s_3, s_4) &= \frac{i}{4} \frac{s_{12}^2 s_{13}^2 + s_{13}^2 s_{14}^2 + s_{14}^2 s_{12}^2}{s_{12} s_{13} s_{14}} \\ &= \frac{i}{16} \frac{(s_{12}^2 + s_{13}^2 + s_{14}^2)^2}{s_{12} s_{13} s_{14}},\end{aligned}\quad (11)$$

by the Mandelstam identity, while on the Yang-Mills side we have

$$\begin{aligned}\mathcal{A}(s_1, s_2, s_3, s_4) &= -i \frac{1}{4} \frac{s_{12}^2 + s_{13}^2 + s_{14}^2}{s_{12} s_{14}}, \quad \text{and} \\ \tilde{\mathcal{A}}(s_1, s_2, s_4, s_3) &= -i \frac{1}{4} \frac{s_{12}^2 + s_{13}^2 + s_{14}^2}{s_{12} s_{13}},\end{aligned}\quad (12)$$

such that

$$\begin{aligned}-is_{12}\mathcal{A}(s_1, s_2, s_3, s_4)\tilde{\mathcal{A}}(s_1, s_2, s_4, s_3) &= \frac{is_{12}}{16} \left(\frac{s_{12}^2 + s_{13}^2 + s_{14}^2}{s_{12} s_{14}} \right) \times \left(\frac{s_{12}^2 + s_{13}^2 + s_{14}^2}{s_{12} s_{13}} \right) \\ &= \frac{i}{16} \frac{(s_{12}^2 + s_{13}^2 + s_{14}^2)^2}{s_{12} s_{13} s_{14}} \\ &= \mathcal{M}(s_1, s_2, s_3, s_4),\end{aligned}\quad (13)$$

It is seen that the KLT relation precisely maps the product of the left/right gauge theory amplitudes into the gravity amplitude.

Next we consider the KLT mapping of a mixed amplitude—the one we will look at is the 2-scalar-2-graviton mapping into the product of two 2-scalar-2-vector amplitudes. Only the partial amplitudes are used where polarizations are contracted with polarizations and momentum with momentum, and for clarity we have chosen to omit the products of the polarizations vectors. The diagrams which need to be considered involve both contact terms and interaction terms. On the gravity side one gets

$$\mathcal{M}(s_1, s_2, h_3, h_4) = \frac{i}{4} \frac{s_{13} s_{14}}{s_{12}}. \quad (14)$$

In a Yang-Mills theory with a mixed matter content there will be different amplitude expressions depending on how we number the particles and these amplitude expressions will relate independently to gravity through KLT. The KLT relation for the above mixed process thus has two independent forms and we hence need expressions for the following three amplitudes:

$$\begin{aligned}\mathcal{A}(s_1, s_2, v_3, v_4) &= -\frac{i}{2} \frac{s_{13}}{s_{12}}, \\ \mathcal{A}(s_1, s_2, v_4, v_3) &= -\frac{i}{2} \frac{s_{14}}{s_{12}}, \quad \text{and} \\ \mathcal{A}(s_1, v_3, s_2, v_4) &= -\frac{i}{2}.\end{aligned}\quad (15)$$

By insertion we immediately see that the two independent gauge amplitude products of the relation are equal to the gravity amplitude

$$\begin{aligned}-is_{12}\mathcal{A}(s_1, s_2, v_3, v_4)\tilde{\mathcal{A}}(s_1, s_2, v_4, v_3) &= -is_{23}\mathcal{A}(s_1, s_2, v_3, v_4)\tilde{\mathcal{A}}(s_1, v_3, s_2, v_4) \\ &= \mathcal{M}(s_1, s_2, h_3, h_4).\end{aligned}\quad (16)$$

Finally, we examine the factorization of the 4-scalar-1-graviton amplitude into the product of two 4-scalar-1-vector amplitudes. This example is the most involved of our checks and is a nontrivial check of KLT. The verification relies on the helicity state notation, see [35] for a nice review. We use the same conventions here. Below we present the results for the case of a helicity (+) graviton or gluon. On the gravity side we have, summing all diagrams which go into this amplitude, that

$$\begin{aligned}\mathcal{M}(s_1, s_2, s_3, s_4, h_5^+) &= (\langle 13 \rangle \langle 42 \rangle [12][34] - \langle 12 \rangle \langle 34 \rangle [13][42]) \\ &\quad \times \frac{(\langle 12 \rangle \langle 34 \rangle \langle 13 \rangle \langle 42 \rangle + \langle 13 \rangle \langle 42 \rangle \langle 14 \rangle \langle 23 \rangle + \langle 14 \rangle \langle 23 \rangle \langle 12 \rangle \langle 34 \rangle)^2}{8 \prod_{i < j} \langle ij \rangle},\end{aligned}\quad (17)$$

where the identity $\langle 13 \rangle \langle 42 \rangle [12][34] - \langle 12 \rangle \langle 34 \rangle [13][42] = 4i\epsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\nu k_3^\rho k_4^\sigma$ can be used.

The Yang-Mills side reads

$$\mathcal{A}(s_1, s_2, s_3, s_4, v_5^+) = \frac{\langle 12 \rangle \langle 34 \rangle \langle 13 \rangle \langle 42 \rangle + \langle 13 \rangle \langle 42 \rangle \langle 14 \rangle \langle 23 \rangle + \langle 14 \rangle \langle 23 \rangle \langle 12 \rangle \langle 34 \rangle}{\sqrt{8} \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}. \quad (18)$$

These amplitudes satisfy the two independent KLT relations for the 5-point functions:

$$\begin{aligned} \mathcal{M}(s_1, s_2, s_3, s_4, h_5^+) &= -is_{12}s_{34}\mathcal{A}(s_1, s_2, s_3, s_4, v_5^+)\tilde{\mathcal{A}}(s_2, s_1, s_4, s_3, v_5^+) - is_{13}s_{24}\mathcal{A}(s_1, s_3, s_2, s_4, v_5^+)\tilde{\mathcal{A}}(s_3, s_1, s_4, s_2, v_5^+) \\ &= -is_{12}s_{35}\mathcal{A}(s_4, s_1, s_2, s_3, v_5^+)\tilde{\mathcal{A}}(s_3, s_4, s_2, s_1, v_5^+) - is_{13}s_{25}\mathcal{A}(s_4, s_1, s_3, s_2, v_5^+)\tilde{\mathcal{A}}(s_2, s_4, s_3, s_1, v_5^+) \end{aligned} \quad (19)$$

as can be seen by insertion (and a fair amount of algebra).

This concludes our set of examples of KLT mapping to order $\mathcal{O}(1)$ in α' in the presence of matter.

B. Effective extensions to $\mathcal{O}(\alpha')$

We now consider the KLT relations in the case of the effective extension of the two theories. The most general gravitational Lagrangian to $\mathcal{O}(\alpha')$ can be written as

$$\begin{aligned} \mathcal{L} = \sqrt{-g} \left[\frac{2R}{\kappa^2} + \frac{1}{2}g^{\mu\nu}D_\mu\phi D_\nu\phi + \alpha'(c_1\kappa^{-2}R_{\mu\nu\alpha\beta}^2 + c_2\kappa^{-2}R_{\mu\nu}^2 + c_3\kappa^{-2}R^2 + c_4\kappa^2D_\mu\phi D^\mu\phi D_\nu\phi D^\nu\phi \right. \\ \left. + c_5\kappa^2\phi D_\mu\phi D_\nu\phi D^\mu D^\nu\phi + c_6\kappa^2\phi^2D_\mu D_\nu\phi D^\mu D^\nu\phi + c_7\phi^2R_{\mu\nu\alpha\beta}^2 + c_8\phi^2R_{\mu\nu}^2 + c_9\phi^2R^2 + \dots \right], \end{aligned} \quad (20)$$

where the ellipses denote higher order terms not necessary for the present analysis.

By a field reparametrization of ϕ and $g_{\mu\nu}$, all coefficients but c_1 , c_4 , and c_7 can be set to any desired value. The terms c_1 and c_7 are left unchanged by such a reparametrization, while c_4 picks up contributions from terms being altered under the reparametrization. Thus, to generate on-shell (4 particle) amplitudes we may limit ourselves to the effective Lagrangian

$$\mathcal{L} = \sqrt{-g} \left[\frac{2R}{\kappa^2} + \frac{1}{2}g^{\mu\nu}D_\mu\phi D_\nu\phi + \alpha'(c_1\kappa^{-2}G_2 + c_4\kappa^2D_\mu\phi D^\mu\phi D_\nu\phi D^\nu\phi + c_7\phi^2R_{\mu\nu\alpha\beta}^2) \right], \quad (21)$$

where $G_2 = R_{\mu\nu\alpha\beta}^2 - 4R_{\mu\nu}^2 + R^2$ is the four dimensional Gauss-Bonnet invariant. This effective extension produces effective corrections to all pure graviton vertices, corrections to all vertices containing four scalars, and a correction to the 2-scalar-2-graviton vertex.²

In the Yang-Mills theory we have a similar situation. We have to include all operators that contain one factor of α' , i.e., those of mass dimension six:

$$\begin{aligned} \mathcal{L} = \text{tr} \left[\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}D_\mu\phi D^\mu\phi + \alpha' \{ a_1(D_\mu F_{\mu\nu})^2 \right. \\ + a_2gF_{\mu\nu}[F_{\nu\lambda}, F_{\lambda\mu}] + a_3g[\phi, D_\mu\phi]D_\nu F_{\mu\nu} \\ + a_4g^2[\phi, F_{\mu\nu}][\phi, F_{\mu\nu}] \\ \left. + a_5g^2[\phi, D_\mu\phi][\phi, D_\mu\phi] + a_6g^2\phi D_\mu D_\mu D_\nu D_\nu\phi \} \right]. \end{aligned} \quad (22)$$

²We use G_2 instead of, e.g., $R_{\mu\nu\alpha\beta}^2$ since that avoids the presence of an effective correction to the graviton propagator in any dimension.

By a field reparametrization we may set a_1 , a_3 , and a_6 to zero while allowing a change in the coefficients a_2 , a_4 , and a_5 . Doing this we end up with the result

$$\begin{aligned} \mathcal{L} = \text{tr} \left[\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}D_\mu\phi D^\mu\phi + \alpha' (a_2gF_{\mu\nu}[F_{\nu\lambda}, F_{\lambda\mu}] \right. \\ + a_4g^2[\phi, F_{\mu\nu}][\phi, F_{\mu\nu}] \\ \left. + a_5g^2[\phi, D_\mu\phi][\phi, D_\mu\phi] \right]. \end{aligned} \quad (23)$$

This effective extension produces corrections to the three-vector vertex, the 2-scalar-2-vector vertex and the four scalar vertex, as well as corrections to vertices not used in this context.

C. Matching of effective operators

The consequences of these extensions are next explored. The KLT relations hold in string theory order by order in $\mathcal{O}(\alpha')$ and it is the expectation from previous investigations Ref. [1,2] that one should expect an effective mapping between the effective field theory operators we have included in the effective Lagrangians. In order to examine this, we have looked into two examples of such possible mappings; namely, the effective extensions of the four scalar amplitudes and the 2-scalar-2-graviton/2-scalar-2-vector amplitudes. Both these amplitude mappings will involve effective operators at order $\mathcal{O}(\alpha')$ and hence relate the effective operators of gravity to those of Yang-Mills through the 4-point KLT relation.

In the case of the four scalar amplitude we have generated the following effective field theory amplitude to order $\mathcal{O}(\alpha')$.

On the gravity side we have

$$\mathcal{M}(s_1, s_2, s_3, s_4) = \frac{i}{16} \left[\frac{(s_{12}^2 + s_{13}^2 + s_{14}^2)^2}{s_{12}s_{13}s_{14}} \right] + 32\alpha' c_4 (s_{12}^2 + s_{13}^2 + s_{14}^2), \quad (24)$$

while on the Yang-Mills side we have

$$\mathcal{A}(s_1, s_2, s_3, s_4) = \frac{i}{4} \frac{-(s_{12}^2 + s_{13}^2 + s_{14}^2) - 6a_5\alpha' s_{12}s_{13}s_{14}}{s_{12}s_{14}}. \quad (25)$$

Matching these amplitudes through KLT at order $\mathcal{O}(\alpha')$ leads to the coefficient relationship

$$c_4 = \frac{3}{8}a_5, \quad (26)$$

which has to hold in order for the mapping to take place.

Looking at the effective amplitudes in the 2-scalar-2-graviton/2-scalar-2-vector case we have on the gravity side

$$\mathcal{M}(s_1, s_2, h_3, h_4) = i \left[\frac{s_{13}s_{14}}{4s_{12}} + \alpha' \left(-\frac{c_1}{4} s_{13}s_{14} + c_7 s_{12}^2 \right) \right], \quad (27)$$

while the necessary Yang-Mills amplitudes are

$$\begin{aligned} \mathcal{A}(s_1, s_2, v_3, v_4) &= i \left[-\frac{s_{13}}{2s_{12}} + \alpha' \left[-\frac{1}{2} a_4 s_{12} + \frac{3}{4} a_2 (s_{13} - s_{14}) \right] \right], \\ \mathcal{A}(s_1, s_2, v_4, v_3) &= i \left[-\frac{s_{14}}{2s_{12}} + \alpha' \left[-\frac{1}{2} a_4 s_{12} + \frac{3}{4} a_2 (s_{14} - s_{13}) \right] \right], \\ \mathcal{A}(s_1, v_3, s_2, v_4) &= i \left[-\frac{1}{2} + \alpha' (-a_4 s_{12}) \right]. \end{aligned} \quad (28)$$

Relating these equations in the same way as in (16), we derive the constraints:

$$c_7 = 0 \quad \text{and} \quad c_1 = 6a_2 \quad \text{and} \quad a_4 = -\frac{3}{2}a_2. \quad (29)$$

The equation $c_1 = 6a_2$ is noted to be exactly what was found in Ref. [1,2] relating the pure effective three amplitude on the gravity side to the three amplitude on the Yang-Mills side. The relationship $a_4 = -\frac{3}{2}a_2$ originates from the requirement that the two “gauge sides” of the KLT relation must be equal. It should be noted that, while the coefficient equations hold to all orders of α' , we cannot from the present analysis determine, if at α'^2 , new coefficient equations generated from the amplitude mapping will constrain the above $\mathcal{O}(\alpha')$ mapping relationships.

IV. DISCUSSION

We have directly shown that the KLT relations also hold for a number of amplitudes involving external scalar legs, and that in such cases, the factorization of gravity amplitudes into Yang-Mills amplitudes is possible. We have also shown through some preliminary examples that an effective extension of these results seems possible, and that this might be a key to gain more insight into the mapping of gravity effective field theory operators into effective Yang-Mills operators. The examples with the four scalar amplitudes and the 2-scalar-2-graviton/2-scalar-2-vector amplitudes clearly suggest that we might gain some important insight including scalars into the effective KLT mapping, although the present examples

should be followed up by some heavier calculations involving more complicated operators.

The amplitude factorization is possible to recast into direct use in the process of calculating gravity amplitudes from much simpler Yang-Mills amplitudes including matter fields.

It is not possible from our present calculation to tell much about an effective generalization of the mapping. In the paper Ref. [2] we replaced the sine functions in the KLT relations with arbitrary polynomials. Such a generalization of the mapping involving matter should be possible in the scalar approach as well, however one has to go to next order in α' , $\mathcal{O}(\alpha'^2)$, to get enough information about how this extension should work.

The starting point for additional investigations in this field could be to look into the effective KLT mapping of the 4-scalar-1-graviton/4-scalar-1-vector amplitudes, since it simultaneously relates all effective gravity and Yang-Mills operators used here.

If the KLT relations have a more fundamental meaning—relating gravity and Yang-Mills theory, the effective mapping of operators—in or without the presence of matter seems to be an adequate starting point for additional theoretical investigations.

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